$$I_{MOD} = \int d^4x\, \mathcal{L}_{MOD}$$

with modified Lagrangian density:

$$\mathcal{L}_{MOD} = \mathcal{L}_M - \frac{1}{4} F_\alpha^{\mu\nu} F_{\alpha\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_\alpha^\mu)(\partial_\nu A_\alpha^\nu)$$

$$- \partial_\mu \omega_\alpha^* \partial^\mu \omega_\alpha + C_{\alpha\beta\gamma} (\partial_\mu \omega_\alpha^*) A_\gamma^\mu \omega_\beta$$

→ Lagrangian is renormalizable
(total dimensionality of products of fields and their derivatives is ≤ 4)

§1.6 BRST Symmetry

Even after we choose a gauge, the path integral still does have a symmetry ... → BRST symmetry

$$I_{MOD} = I_{EFF} + I_{GH} = \int d^4x\, \mathcal{L}_{MOD},$$

$$\mathcal{L}_{MOD} = \mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha + \omega_\alpha^* \Delta_\alpha,$$

where $\Delta_\alpha(x) \equiv \int d^4x\, \mathcal{F}_{\alpha x, \beta y}[A, \psi] \omega_\beta(y)$

for the choice $B[f] = \exp\left(-\frac{i}{2\xi} \int d^4x\, f_\alpha f^\alpha\right)$

rewrite $B[f] = \int \left[\prod_{\alpha, x} dh_\alpha(x)\right] \exp\left[\frac{i\xi}{2} \int d^4x\, h_\alpha h_\alpha\right] \exp\left[i \int d^4x\, f_\alpha h_\alpha\right]$

→ new modified action: $I_{NEW} = \int d^4x \left(\mathcal{L} + \omega_\alpha^* \Delta_\alpha + h_\alpha f_\alpha + \frac{1}{2} \xi h_\alpha h_\alpha\right)$

→ not gauge invariant but enjoys the following symmetry:

let θ be an infinitesimal constant with
$[\theta, \omega_\alpha]_+ = [\theta, \omega_\alpha^*]_+ = 0$ (anti-commutator vanishes)

Then

$$(1)\quad \begin{cases} \delta_\theta \psi = i t_\alpha \theta \omega_\alpha \psi, \\ \delta_\theta A_{\alpha\mu} = \theta D_\mu \omega_\alpha = \theta[\partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma], \\ \delta_\theta \omega_\alpha^* = -\theta h_\alpha, \\ \delta_\theta \omega_\alpha = -\frac{1}{2}\theta C_{\alpha\beta\gamma}\omega_\beta\omega_\gamma, \\ \delta_\theta h_\alpha = 0. \end{cases}$$

keeps the action invariant.

Let us see how this works:

a) transformation (1) is "nilpotent",
namely for $\delta_\theta F[\psi, A, \omega, \omega^*] \equiv \theta s F$

$$\delta_\theta (sF) = 0 \iff s(sF) = 0$$

Let us check this for matter fields

$$\delta_\theta s\psi = i t_\alpha \delta_\theta(\omega_\alpha \psi)$$

$$= -\frac{1}{2} i C_{\alpha\beta\gamma} t_\alpha \theta \omega_\beta \omega_\gamma \psi - t_\alpha t_\beta \omega_\alpha \theta \omega_\beta \psi$$

$$= -\frac{1}{2} i C_{\alpha\beta\gamma} t_\alpha \theta \omega_\beta \omega_\gamma \psi + \underbrace{t_\alpha t_\beta \theta \omega_\alpha \omega_\beta \psi}_{= \frac{1}{2}[t_\alpha, t_\beta]\theta\omega_\alpha\omega_\beta\psi}$$

$$= 0$$

$\rightarrow ss\psi = 0$

Next, acting on a gauge field, we have

$$\delta_\theta s A_{\alpha\mu} = \delta_\theta D_\mu \omega_\alpha$$

$$= \partial_\mu \delta_\theta \omega_\alpha + C_{\alpha\beta\gamma} \delta_\theta A_{\beta\mu} \omega_\gamma + C_{\alpha\beta\gamma} A_{\beta\mu} \delta_\theta \omega_\gamma$$

$$= \theta\Big(-\frac{1}{2} C_{\alpha\beta\gamma} \partial_\mu(\omega_\beta \omega_\gamma) + C_{\alpha\beta\gamma}(\partial_\mu \omega_\beta)\omega_\gamma$$

$$+ C_{\alpha\beta\gamma} C_{\beta\delta\varepsilon} A_{\delta\mu} \omega_\varepsilon \omega_\gamma - \frac{1}{2} C_{\alpha\beta\gamma} C_{\gamma\delta\varepsilon} A_{\beta\mu} \omega_\delta \omega_\varepsilon \Big)$$

$$= \theta\Big(\frac{1}{2} C_{\alpha\beta\gamma}(\partial_\mu \omega_\beta)\omega_\gamma + \frac{1}{2} C_{\alpha\beta\gamma}(\partial_\mu \omega_\gamma)\omega_\beta$$

$$- C_{\alpha\beta\gamma} C_{\gamma\delta\varepsilon} A_{\delta\mu} \omega_\varepsilon \omega_\beta - \frac{1}{2} C_{\alpha\beta\gamma} C_{\gamma\delta\varepsilon} A_{\beta\mu} \omega_\delta \omega_\varepsilon \Big)$$

$$= 0$$

$\rightarrow ss A_{\alpha\mu} = 0$

Also $ss\omega^*_\alpha = 0$ and $ssh^*_\alpha = 0$ (trivial)

Finally,

$$\delta_\theta s\omega_\alpha = -\frac{1}{2} C_{\alpha\beta\gamma} \delta_\theta(\omega_\beta \omega_\gamma)$$

$$= \frac{1}{4}\theta\left(C_{\alpha\beta\gamma} C_{\beta\delta\varepsilon} \omega_\delta \omega_\varepsilon \omega_\gamma - C_{\alpha\beta\gamma} C_{\gamma\delta\varepsilon} \omega_\beta \omega_\delta \omega_\varepsilon\right)$$

$$= \frac{1}{2}\theta C_{\alpha\beta\gamma} C_{\gamma\delta\varepsilon} \omega_\delta \omega_\varepsilon \omega_\beta$$

$= 0$ by Jacobi-identity

$\rightarrow ss\omega_\alpha = 0$

Now consider a product of two fields ϕ_1 and ϕ_2

$$\rightarrow \delta_\theta(\phi_1 \phi_2) = \theta(s\phi_1)\phi_2 + \phi_1 \theta(s\phi_2) = \theta\left[(s\phi_1)\phi_2 \pm \phi_1 s\phi_2\right]$$

where the sign $\pm$ is plus for ϕ_1 bosonic and $-$ for ϕ_1 fermionic.

$\rightarrow\ s(\phi_1 \phi_2) = (s\phi_1)\phi_2 \pm \phi_1 s\phi_2$

Since $\delta_\theta(s\phi_1) = \delta_\theta(s\phi_2) = 0$, we get

$\delta_\theta s(\phi_1 \phi_2) = (s\phi_1)\theta(s\phi_2) \pm \theta(s\phi_1)(s\phi_2)$

But: $s\phi$ has opposite statistics to ϕ_1

$\rightarrow\ \delta_\theta s(\phi_1 \phi_2) = \theta\left[\mp(s\phi_1)(s\phi_2) \pm (s\phi_1)(s\phi_2)\right] = 0$

Continuing the same argument, one sees that

$$\delta_\theta\, s(\phi_1 \phi_2 \phi_3 \dots) = 0$$

Any functional $F[\phi]$ can be written as a sum of products of fields (with integrals).

Thus $\delta_\theta sF[\phi] = \theta ss F[\phi] = 0$.

b) Invariance of action:

For $F = F[A, \psi]$, δ_θ acts like a gauge trf. with infinitesimal gauge parameter

$$\lambda_\alpha(x) = \theta\omega_\alpha(x)$$

$$\rightarrow\ \delta_\theta \int d^4x\, \mathcal{L} = 0$$

Acting on other terms, we get

$$\delta_\theta f_\alpha[x; A, \psi] = \int \frac{\delta f_\alpha[x; A_\lambda, \psi_\lambda]}{\delta\lambda^\beta(y)}\Big|_{\lambda=0} \theta\omega_\beta(y) d^4y$$

$$= \theta \int \mathcal{F}_{\alpha x, \beta y}[A, \psi]\omega_\beta(y) d^4y = \theta\Delta_\alpha(x; A, \psi, \omega)$$

Also recall: $\delta_\theta \omega_\alpha^* = -\theta h_\alpha$ and $\delta_\theta h_\alpha = 0$

$\longrightarrow \omega_\alpha^* \Delta_\alpha + h_\alpha f_\alpha + \frac{1}{2}\xi h_\alpha h_\alpha$

$= -s\left(\omega_\alpha^* f_\alpha + \frac{1}{2}\xi\omega_\alpha^* h_\alpha\right)$

or in other words

$$I_{NEW} = \int d^4x\, \mathcal{L} + s\Psi,$$

where $\Psi \equiv -\int d^4x \left(\omega_\alpha^* f_\alpha + \frac{1}{2}\xi \omega_\alpha^* h_\alpha\right)$

$\longrightarrow$ nilpotence of BRST trf. tells us that $s\Psi$ is also invariant!

<u>Remark:</u>

Physical content of a gauge theory (encoded in $\int d^4x\, \mathcal{L}$) is contained in $\mathrm{Ker}\,\delta_\theta / \mathrm{Im}\,\delta_\theta$!

$$\underset{\int d^4x \mathcal{L} + s\Psi}{\mathrm{Ker}\,\delta_\theta} \Big/ \underset{s\Psi}{\mathrm{Im}\,\delta_\theta}$$

$\longrightarrow$ form elements of the "cohomology" of δ_θ.

Physical matrix elements should be invariant under change $\delta\Psi$ in Ψ:

(1) $\delta\langle\alpha|\beta\rangle = i\langle\alpha|\delta I_{NEW}|\beta\rangle = i\langle\alpha|s\delta\Psi|\beta\rangle$

Introduce fermionic BRST "charge" Q:

$$\delta_\theta \Phi = i[\theta Q, \Phi] = i\theta[Q,\Phi]_\mp \quad \text{or} \quad [Q,\Phi]_\mp = is\Phi$$

sign is being $-$ or $+$ for bosonic or fermionic Φ.

$\rightarrow$ nilpotence gives $0 = -ss\Phi = [Q, [Q, \Phi]_\pm]_\pm = [Q^2, \Phi]_-$.

$\rightarrow$ $Q^2 = 0$.

Thus eq. (2) becomes

$$\delta \langle\alpha|\beta\rangle = \langle\alpha|[Q, \delta\Psi]|\beta\rangle$$

$$\rightarrow \quad \langle\alpha| Q = Q|\beta\rangle = 0 . \qquad (*)$$

"Independent" physical states correspond to states in the kernel of Q modulo the image of Q.

$\rightarrow$ elements of Q-cohomology!

Example:

Consider pure QED and take $f = \partial_\mu A^\mu$

$\rightarrow$ BRST trf. is

$$sA_\mu = \partial_\mu \omega, \qquad s\omega^* = \partial_\mu A^\mu/\xi, \qquad s\omega = 0 \qquad (3)$$

Using normal mode expansions

$$A^\mu(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[a^\mu(\vec{p})e^{ip\cdot x} + a^{\mu *}(\vec{p})e^{-ip\cdot x}\right],$$

$$\omega(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[c(\vec{p})e^{ip\cdot x} + c^*(\vec{p})e^{-ip\cdot x}\right],$$

$$\omega^*(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[b(\vec{p})e^{ip\cdot x} + b^*(p)e^{-ip\cdot x}\right].$$

Matching coefficients on both sides of (3), we get:

$$[Q, a^\mu(\vec{p})]_- = p^\mu c(\vec{p}), \quad [Q, a^{\mu *}(\vec{p})]_- = p^\mu c^*(\vec{p}),$$

$$[Q, b(\vec{p})]_+ = p^\mu a_\mu(\vec{p})/\xi, \quad [Q, b^*(\vec{p})]_+ = p^\mu a_\mu^*(\vec{p})/\xi,$$

$$[Q, c(\vec{p})]_+ = [Q, c^*(\vec{p})]_+ = 0$$

Consider any physical state $|\psi\rangle$ with $Q|\psi\rangle = 0$

$\rightarrow$ $|e, \psi\rangle = e_\mu a^{\mu *}(\vec{p})|\psi\rangle$ with one additional photon then satisfies

$$Q|e, \psi\rangle = 0 \quad \text{if } e_\mu p^\mu = 0$$

also $|\psi\rangle' \equiv b^*(\vec{p})|\psi\rangle$ satisfies

$$Q|\psi\rangle' = p^\mu a_\mu^*(\vec{p})|\psi\rangle/\xi$$

$\Rightarrow$ $|e + \alpha p, \psi\rangle = |e, \psi\rangle + \xi\alpha\, Q|\psi\rangle'$

$\rightarrow$ e^μ is physically equivalent to $e^\mu + \alpha p^\mu$ (usual gauge invariance)

But $Q b^*(\vec{p})|\psi\rangle = p^\mu a_\mu^*(\vec{p})|\psi\rangle \neq 0$,

$\rightarrow$ $b^*|\psi\rangle$ is not in physical Hilbert space (*)

Also $c^*(p)|\psi\rangle = Q e_\mu a^{*\mu}(\vec{p})|\psi\rangle / e.p$

$\rightarrow$ BRST exact $\rightarrow$ also not in Hilbert space

$\rightarrow$ Thus the physical Hilbert space is <u>free</u> of ghosts and antighosts!